Week 13
NOV 29

Lecture

I Stokes' thm & Consequences
II Stokes' thm, proof
III Divergence thm & Consequences
IV Some Remarks • Vector identities
• direction cosines
V Multi-connected regions

I Stokes' thm & Consequences (Cont'd)

Let's look at one more example.

e.g. 1 Consider the cone $z=(x^2+y^2)^{1/2}$ cut at $z=2$ which is regarded as a surface with normal pointing inward. The boundary curve C : $(x,y,2)$, $(x,y)\in D_2$, is oriented in anticlockwise way. Evaluate

$$\oint_C ((x^2-y)\hat{i}+4z\hat{j}+x^2\hat{k})\cdot d\vec{r}.$$

The orientation of S and C are consistent. By Stokes' theorem

$$\oint_C ((x^2-y)\hat{i}+4z\hat{j}+x^2\hat{k})\cdot d\vec{r}$$

$$=\iint_S \nabla\times\vec{F}\cdot\hat{n}\,d\sigma$$

$$=\iint_{D_2} \nabla\times\vec{F}\cdot\vec{r}_x\times\vec{r}_y\,dA(x,y).$$

$$\nabla\times\vec{F}=\begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ x^2-y & 4z & x^2 \end{vmatrix} = -4\hat{i}-2x\hat{j}+\hat{k}.$$

$\vec{r}_x \times \vec{r}_y = (-f_x, -f_y, 1)$ when $(x,y) \mapsto (x, y, f(x,y))$ graph.

Here $f(x,y) = \sqrt{x^2+y^2}$,

$$= \left(\frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1 \right)$$

$$\therefore \iint_{D_2} \nabla\times\vec{F}\cdot \vec{r}_x\times\vec{r}_y \, dA(x,y) = \iint_{D_2} (-4\hat{i} - 2x\hat{j} + \hat{k})\left(\frac{-x}{\sqrt{}}\hat{i} + \frac{-y}{\sqrt{}}\hat{j} + \hat{k}\right) dA$$

$$= \int_0^{2\pi}\int_0^2 (-4\hat{i} - 2r\cos\theta\,\hat{j} + \hat{k})(-\cos\theta\,\hat{i} - \sin\theta\,\hat{j} + \hat{k})\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 (4r\cos\theta + 2r^2\cos\theta\sin\theta + r)\, dr\, d\theta$$

$$\vdots$$

$$= 4\pi.$$

$$\therefore \oint_C ((x^2-y)\hat{i} + 4z\hat{j} + x^2\hat{k})\cdot d\vec{r} = 4\pi.$$

A smarter approach is to observe that C is also the boundary of the surface $S_1 = (x, y, 2)$, $(x,y) \in D_2$. With normal $\hat{n} = (0,0,1)$, its orientation is consistent with C.

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \iint_{S_1} \nabla\times\vec{F}\cdot \vec{r}_x\times\vec{r}_y \, dA, \quad \vec{r}_x = (1,0,0), \quad \vec{r}_y = (0,1,0), \quad \vec{r}_x\times\vec{r}_y = (0,0,1)$$

$$= \iint_{D_2} (-4\hat{i} - 2x\hat{j} + \hat{k})\cdot\hat{k}\, dA$$

$$= \iint_{D_2} dA$$

$$= 4\pi \quad \#$$

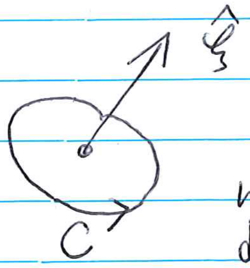
There are two more applications of Stokes' theorem.
The first is the following theorem.

<u>Theorem</u> Let $\vec{F}$ be a vector field in a simply-connected region in space. Then $\vec{F}$ is conservative if and only if it satisfies the component test.

See Assignment 11 for details.

The next application is to motivate the definition of the curl density (or circulation density).

Let $\hat{\xi}$ be a direction (ie a unit vector) at a point $\vec{x} = x\hat{i} + y\hat{j} + z\hat{k}$ in $\mathbb{R}^3$.



We consider a disk of radius a centered at $\vec{x}$ whose chosen normal $\hat{n} = \hat{\xi}$. Take the anticlockwise direction on its boundary C_a so that

$$\iint_{D_a} \nabla \times \vec{F} \cdot \hat{n}\, d\sigma = \oint_{C_a} \vec{F} \cdot d\vec{r} = \text{the circulation of } \vec{F} \text{ around } C_a$$

$$\therefore \frac{1}{|D_a|} \oint_{C_a} \vec{F} \cdot d\vec{r} \to \nabla \times \vec{F} \cdot \hat{n} = \nabla \times \vec{F} \cdot \hat{\xi} \quad \text{as } a \to 0.$$

It suggests to define the curl of $\vec{F}$ to be $\nabla \times \vec{F}$ so that $\nabla \times F \cdot \hat{\xi}$ gives the circulation density of $\vec{F}$ at the direction $\hat{\xi}$.
When $\hat{\xi} = (0,0,1)$, $\nabla \times \vec{F} = N_x - M_y$ reduces to the plane case.

(II) Now we proof Stokes theorem

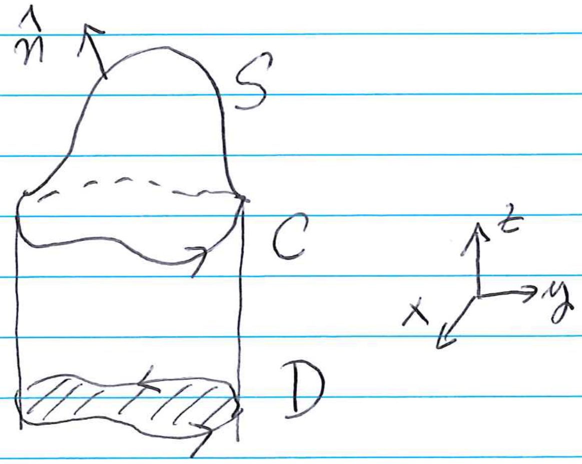
$$\iint_S \nabla\times\vec{F}\cdot\hat{n}\,d\sigma = \oint_C \vec{F}\cdot d\vec{r}.$$

Let's assume

$$S=\{(x,y,f(x,y)),\ (x,y)\in D\}$$

$$C=\{(x(t),y(t),f(x(t),y(t)))\}$$ where

$t \longmapsto (x(t),y(t))$ parametrize the boundary of D, C_1.



$\vec{F}=M\hat{i}+N\hat{j}+P\hat{k}$

$\nabla\times\vec{F}=(P_y-N_z)\hat{i}+(-P_x+M_z)\hat{j}$
$\quad +(N_x-M_y)\hat{k},$

$(x,y)\longmapsto(x,y,f(x,y))$ parametrizes S

$r_x\times r_y=(-f_x,-f_y,1)$ (pointing up)

$$\iint_S \nabla\times\vec{F}\cdot\hat{n}\,d\sigma=\iint_D \nabla\times\vec{F}(x,y,f)\cdot(-f_x\hat{i}+-f_y\hat{j}+\hat{k})\,dA(x,y)$$

$$=\iint_D (P_y-N_z)(-f_x)+(-P_x+M_z)(-f_y)+(N_x-M_y)\,dA \quad \text{——} (1)$$

Next, $$\oint_C \vec{F}\cdot d\vec{r}=\oint_C M\,dx+N\,dy+P\,dz$$

$$= \int_a^b M(x(t), y(t), f(x(t), y(t)))\, x'(t) + N(x(t), y(t), f(x(t), y(t)))\, y'(t)$$
$$+ P(x(t), y(t), f(x(t), y(t)))(f_x x' + f_y y')\, dt \qquad — ②$$

Note that $z(t) = f(x(t), y(t))$, so $z'(t) = f_x x' + f_y y'$.

Now, set
$$\tilde{M}(x,y) = M(x, y, f(x,y)) + P(x, y, f(x,y))\, f_x(x,y)$$
and
$$\tilde{N}(x,y) = N(x, y, f(x,y)) + P(x, y, f(x,y))\, f_y(x,y).$$

$\tilde{M}\hat{i} + \tilde{N}\hat{j}$ is a v.f. on D and
$$\oint_{C_1} \tilde{M}dx + \tilde{N}dy = ②.$$

By Green's thm,
$$\oint_{C_1} \tilde{M}dx + \tilde{N}dy = \iint_D (\tilde{N}_x - \tilde{M}_y)\, dA. \qquad — ③$$

$$\tilde{N}_x = N_x + N_z f_x + (P_x + P_z f_x) f_y + P f_{yx}$$
$$\tilde{M}_y = M_y + M_z f_y + (P_y + P_z f_y) f_x + P f_{xy}$$

so
$$\tilde{N}_x - \tilde{M}_y = N_x - M_y + N_z f_x - M_z f_y + P_x f_y - P_y f_x$$
$$= (P_y - N_z)(-f_x) + (-P_x + M_z)(-f_y) + N_x - M_y. \qquad — ④$$

Comparing ① and ④,
$$\iint_S \nabla\times\vec{F}\cdot\hat{n}\, d\sigma = \oint_C \vec{F}\cdot d\vec{r}. \quad \#$$

(III) Divergence theorem.
this is the Green's theorem in space.

Theorem Let Ω be a region bounded by a closed surface in space. For a smooth v.f F in Ω,

$$\iiint_{\Omega} \nabla\cdot F\, dV = \oiint_{S} \vec{F}\cdot\hat{n}\, d\sigma ,$$

where

$$\operatorname{div} F = \nabla\cdot F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z},$$

and $\hat{n}$ is the outward unit normal at S.

Pf: Claim

$$\iiint_{\Omega} \frac{\partial M}{\partial x}\, dV = \oiint_{S} M n_1\, d\sigma$$

$$\iiint_{\Omega} \frac{\partial N}{\partial y}\, dV = \oiint_{S} N n_2\, d\sigma$$

$$\iiint_{\Omega} \frac{\partial P}{\partial z}\, dV = \oiint_{S} P n_3\, d\sigma .$$

We only prove the last one. Let's assume

$$S = S_1 \cup S_2 ,$$

$$S_1 = \{ (x, y, f_1(x,y)), (x,y)\in D \} \qquad f_1(x,y) \le f_2(x,y)$$

$$S_2 = \{ (x, y, f_2(x,y)), (x,y)\in D \}$$

then

$$\iiint_{\Omega} \frac{\partial P}{\partial z}\, dV = \iint_{D} \int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial P}{\partial z}\, dz\, dA(x,y)$$

$$= \iint_{D} P(x,y,f_2(x,y)) - P(x,y,f_1(x,y))\, dA(x,y). \quad \text{— ①}$$

On the other hand, $\hat{n} = \frac{(-f_{2x}, -f_{2y}, 1)}{\sqrt{1+f_{2x}^2+f_{2y}^2}}$ on S_2

$\hat{n} = \left(\frac{+f_{1x}, +f_{1y}, -1}{\sqrt{1+f_{1x}^2+f_{1y}^2}}\right)$ on S_1.

$$\therefore \oiint_{S} P n_3\, d\sigma = \iint_{S_1} P n_3\, d\sigma + \iint_{S_2} P n_3\, d\sigma$$

$$= \iint_{S_1} P(x,y,f_1(x,y))(-1)\, dx\, dy + \iint_{S_2} P(x,y,f_2(x,y))(1)\, dx\, dy$$

$= ①$, done #

We use the div. thm to obtain the flux density (or the divergence)

$$\frac{1}{|\Omega|} \oiint_{S} F\cdot\hat{n}\, d\sigma = \frac{1}{|\Omega|} \iiint_{\Omega} \nabla\cdot F\, dV$$

$$\to \nabla\cdot F(\vec{x}) \quad \text{as } a \to 0$$

(take Ω to be the ball of radius a around $\vec{x}$.)

IV Two Remarks.

We note the following two vector identities.

~ $\nabla \times \nabla g = \vec{0}$ (Gradient v.f.'s are curl free).

~ $\nabla \cdot \nabla \times \vec{F} = 0$ (Curl v.f's are divergence free)

Pf: $\nabla g = g_x \hat{i} + g_y \hat{j} + g_z \hat{k}$

$$\nabla \times \nabla g = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ g_x & g_y & g_z \end{vmatrix}$$

$$= (g_{zy} - g_{yz})\hat{i} - (g_{zx} - g_{xz})\hat{j} + (g_{yx} - g_{xy})\hat{k}$$

$$= \vec{0}.$$

$$\nabla \times \vec{F} = (P_y - N_z)\hat{i} - (P_x - M_z)\hat{j} + (N_x - M_y)\hat{k}$$

$$\nabla \cdot \nabla \times \vec{F} = (P_y - N_z)_x - (P_x - M_z)_y + (N_x - M_y)_z$$

$$= P_{yx} - N_{zx} - P_{xy} + M_{zy} + N_{xz} - M_{yz}$$

$$= 0.$$

The converse issue: when $\vec{F}, \vec{H}$ are defined in $\mathbb{R}^3$,

~ If $\nabla \times \vec{F} = \vec{0}$, then there is some g s.t. $\vec{F} = \nabla g$.

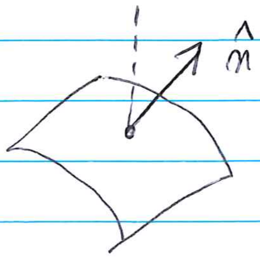
~ If $\nabla \cdot \vec{H} = 0$, then there is some $\vec{F}$ s.t. $\vec{H} = \nabla \times \vec{F}$,

the first statement works on any simply-connected region.
Just the same as $\vec{F}$ is conservative iff the component test is satisfied.
The second statement is more difficult, we'll not consider it.

Next, we explain the notation

$$\iint_S \vec{F}\cdot\hat{n}\, d\sigma = \iint_S M\, dy\, dz + N\, dx\, dz + P\, dx\, dy,$$

for $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$.



$\hat{n} = \dfrac{(-f_x, -f_y, 1)}{\sqrt{1+f_x^2+f_y^2}}$, $\therefore\ n_3 = \dfrac{1}{\sqrt{1+f_x^2+f_y^2}}$.

We have $\quad \vec{F}\cdot\hat{n}\, d\sigma = (Mn_1 + Nn_2 + Pn_3)\, d\sigma$.

claim: $\quad dy\, dz = n_1\, d\sigma$

$\quad dx\, dz = n_2\, d\sigma$

$\quad dx\, dy = n_3\, d\sigma$.

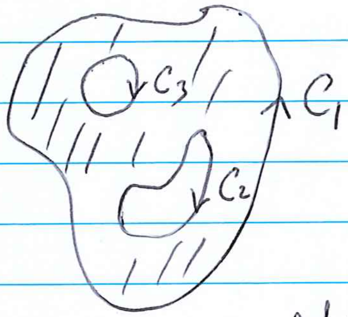
Let's prove the last one:

$$n_3\, d\sigma = \frac{1}{\sqrt{1+f_x^2+f_y^2}} \times \sqrt{1+f_x^2+f_y^2}\, dx\, dy = dx\, dy.$$

$\uparrow$

~ This was omitted from class ~

(V) Multi-connected Regions

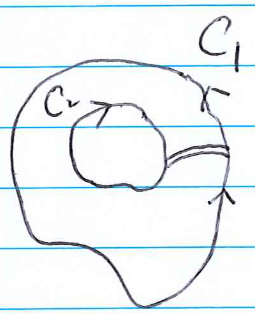


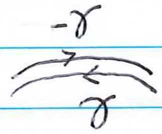
<u>Green's Theorem</u>. Let D be bounded by C_1 outside and $C_2, \ldots C_n$ inside. Let $\vec{F}$ be a v.f. in D. Then

$$\iint_D (N_x - M_y)\, dA(x,y) = \sum_{j=1}^{n} \oint_{C_j} M dx + N dy,$$

where C_1 is oriented in anticlockwise, and $C_2, \ldots C_3$ in clockwise direction.

Idea of Proof. Cut up C_1, C_2 into a single closed curve $C_1 + \gamma + C_2 + (-\gamma)$. Then





$$\oint_{C_1} + \oint_{C_2} M dx + N dy$$

$$= \oint_{C_1 + \gamma + C_2 + (-\gamma)} M dx + N dy$$

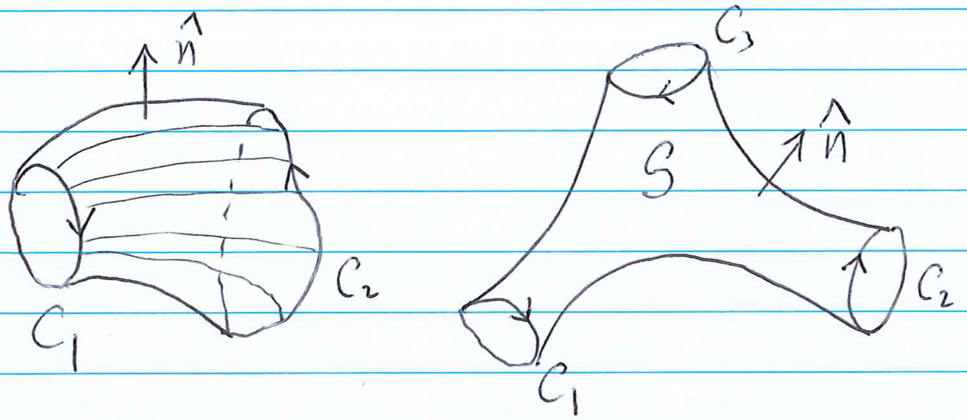
$$= \iint_D (N_x - M_y)\, dA(x,y)$$

By similar cut-up trick,

<u>Stokes' thm</u> Let S be a surface in space with boundary $C_1, \ldots C_n$. Then

$$\iint_S \nabla \times \vec{F} \cdot \hat{n}\, d\sigma = \sum_{j=1}^{n} \oint_{C_j} \vec{F} \cdot d\vec{r},$$

where the orientations of S and C_j's are consistent.



Divergence thm Let Ω be a region in space bounded by closed surfaces $S_1, \dots, S_n$. Then for v.f $\vec{F}$ in Ω,

$$\iiint_{\Omega} \nabla \cdot \vec{F}\, dV = \sum_{j=1}^{n} \oiint_{S_j} \vec{F} \cdot \hat{n}\, d\sigma \quad , \qquad \text{where}$$

$\hat{n}$ is the outer unit normal at S_j.

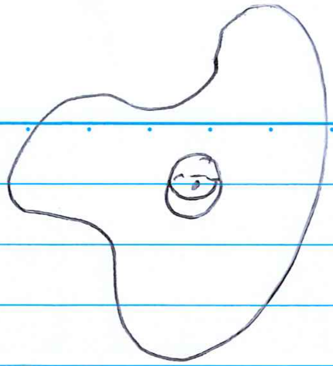
An application : Gauss' law.

A point charge of charge q located at the origin generates a v.f. (electric field)

$$\vec{E} = \frac{q}{4\pi\varepsilon_0} \frac{\vec{x}}{|\vec{x}|^3} \,, \qquad \varepsilon_0 > 0 \text{ some constant.}$$

Gauss' Law. The flux through every closed surface enclosing the origin is the same and equals to

$$q/\varepsilon_0 \,.$$



Note that $\vec{E}$ is not defined at the origin.

Let B_a be the ball centered at the origin with radius a, and S_a its boundary sphere.

$a > 0$ small, that B_a is enclosed inside S.

Let Ω_a be the region bounded by S and S_a. Observe $\vec{E}$ is a smooth v.f. in Ω_a. We can apply Divergence thm to get

$$\oiint_S \vec{E}\cdot\hat{n}\,d\sigma + \oiint_{S_a} \vec{E}\cdot\hat{n}\,d\sigma = \iiint_{\Omega_a} \nabla\cdot\vec{E}\,dV$$

By a direct computation, $\nabla\cdot\vec{E} = 0 \quad \forall \vec{x} \neq \vec{0}$.

$$\therefore \oiint_S \vec{E}\cdot\hat{n}\,d\sigma = -\oiint_{S_a} \vec{E}\cdot\hat{n}\,d\sigma.$$

At $\vec{x} \in S_a$, its outer normal (w.r.t. Ω_a) is $\frac{-\vec{x}}{|\vec{x}|}$, so

$$-\vec{E}\cdot\hat{n} = -\frac{q}{4\pi\varepsilon_0}\frac{\vec{x}}{|\vec{x}|^3}\cdot\frac{-\vec{x}}{|\vec{x}|}$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{a^2}{a^4}, \qquad |\vec{x}| = a$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{a^2}$$

$$\therefore \oiint_S \vec{E}\cdot\hat{n}\,d\sigma = \frac{q}{4\pi\varepsilon_0}\frac{1}{a^2}\oiint_{S_a} d\sigma = \frac{q}{\varepsilon_0} \quad \#$$